WEAK FUNCTORIALITY OF COHEN-MACAULAY ALGEBRAS

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ABSTRACT. We prove the weak functoriality of (big) Cohen-Macaulay algebras, which controls the whole skein of "homological conjectures" in commutative algebra [10][13]; namely, for any local homomorphism $R \to R'$ of complete local domains, there exists a compatible homomorphism between some Cohen-Macaulay R-algebra and some Cohen-Macaulay R'-algebra.

When R contains a field, this is already known [13, 3.9]. When R is of mixed characteristic, our strategy of proof is reminiscent of G. Dietz's refined treatment [7] of weak functoriality of Cohen-Macaulay algebras in characteristic p; in fact, developing a "tilting argument" due to K. Shimomoto, we combine the perfectoid techniques of [1][2] with Dietz's result.

1. INTRODUCTION.

1.1. If non-Noetherian rings have found their place in standard commutative algebra, it is not so much as pathological examples or products of an inexorable generalization process, but rather as invaluable auxiliaries when the resources of classical Cohen-Macaulay theory fail. Indeed, "unwanted relations" between parameters of a Noetherian local ring R which do not form a regular sequence may still be trivialized in some big, *i.e.* possibly non-Noetherian, R-algebra.

This accounts for the importance of the existence of (big) Cohen-Macaulay R-algebras in commutative algebra (*cf.* [10][13][14][3] for some overviews; in the sequel, we drop the epithet big). Their existence has been known and used for a long time when R contains a field, but it was proven only recently in mixed characteristic [2], using methods from p-adic Hodge theory (perfectoid spaces).

For more sophisticated applications to the "homological conjectures" in commutative algebra, more is required: a weakly functorial behaviour of these Cohen-Macaulay algebras; namely, it is expected that for any local homomorphism $R \overset{f}{\to} R'$ of complete local domains, there exists a compatible homomorphism between some Cohen-Macaulay R-algebra and some Cohen-Macaulay R'-algebra. Again, this has been known for a long time when R contains a field [13]; in characteristic p, it actually suffices to consider a lifting $R^+ \overset{f^+}{\to} R'^+$ of f to the absolute integral closures [12]. In this paper, we establish this weak functoriality in general:

1.1.1. **Theorem.** *Any local homomorphism $R\to R'$ of complete Noetherian local domains fits into a commutative square*

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ C & \longrightarrow & C' \end{array} \tag{1.1}$$

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where C and C' are Cohen-Macaulay algebras for R and R' respectively.

1.2. In the remaining case to be treated, *i.e.* when R is of mixed characteristic, we prove the following more precise result:

1.2.1. **Theorem.** *Any local homomorphism $f : R \to R'$ of complete Noetherian local domains, with R of mixed characteristic $(0, p)$, fits into a commutative square*

(1.2)

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ \downarrow & & \downarrow \\ R^+ & \xrightarrow{f^+} & R'^+ \\ \downarrow & & \downarrow \\ C & \longrightarrow & C' \end{array}$$

where R^+ and R'^+ are the absolute integral closures of R and R' respectively, C and C' are Cohen-Macaulay algebras for R and R' respectively, C is $\hat{\mathcal{K}}^o$-perfectoid, and C' is $\hat{\mathcal{K}}^o$-perfectoid if R' is of mixed characteristic $(0, p)$ (resp. is perfect if R' is of characteristic p). Moreover, f^+ can be given in advance.

Here $\hat{\mathcal{K}}^o$ is the perfectoid ring obtained from the Witt ring of the algebraic closure of the residue field of R' by adjoining p^{th}-power roots of p and completing; "f^+ can be given in advance" means that for any choice of the upper commutative square, one can form a commutative lower square as indicated. The compatibility with f^+ is not just a refinement of weak functoriality of Cohen-Macaulay algebras, but plays a crucial role in the proof.

1.3. Our strategy is inspired by Dietz's refined weak functoriality of Cohen-Macaulay algebras in characteristic p. Using Cohen factorizations, one reduces to the case of a surjection, in which case one proves a stronger version of weak functoriality where C is given in advance: this allows to compose or decompose the morphism f, hence to reduce to the case when f is a quotient map by a prime ideal of height one (and R is normal)[1].

To deal with this special case in the char. p situation, Dietz uses a subtle Frobenius argument. In order to transpose it somehow to mixed characteristic, we take advantage of a remarkable insight of K. Shimomoto [20], who combined the perfectoid techniques from [1][2] with Dietz's results (through tilting) to show the existence of a *perfectoid* Cohen-Macaulay R-algebra C, so that perfectoid techniques can be applied to C itself.

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I am very grateful to O. Gabber for his critical remarks, notably for pointing out an unclear non-(almost)-triviality issue in a previous version and for suggesting the use of solid closure theory to settle this issue (*cf.* 2.5.1 (2); he also proposed a different solution to this technical problem).

Finally, I am indebted to the referees' very careful reading and useful advice.

[1] in this special situation (height one specialization), weak functoriality has already been established in [9] in mixed characteristic, using perfectoid methods after [2], and is used to get important corollaries such as the vanishing conjecture on Tor's. The main progress in the present paper in this special situation lies, as said above, in the fact that C may be given in advance.

2. Review of absolute integral closure, (almost) perfectoid algebras and (almost) Cohen-Macaulay algebras.

2.1. **Weak functoriality of absolute integral closure.** (See also [14, 3]). Let R be a domain with fraction field K, K^+ an algebraic closure of K, and R^+ the integral closure of R in K^+ (so that $R^+ \cap K$ is the normalization of R). In this situation, one may identify the absolute automorphism group $G_K = \mathrm{Aut}_K K^+$ with the group of R-automorphisms of R^+.

Let R' be another domain with fraction field K' of characteristic 0, K'^+ an algebraic closure of K, and R'^+ the integral closure of R' in K'^+.

2.1.1. **Proposition.** (1) *Any homomorphism $R \xrightarrow{f} R'$ lifts to a homomorphism $R^+ \xrightarrow{f^+} R'^+$. If f is injective (resp. surjective), so is f^+.*

(2) *Assume that f is injective or R is normal. Then f^+ is unique up to precomposition by an element of G_K.*

(3) *Assume that R is normal. Any factorization $R \xrightarrow{f_1} R_1 \to \cdots \to R_{n-1} \xrightarrow{f_n} R'$ of f lifts to a factorization $R^+ \xrightarrow{f_1^+} R_1^+ \to \cdots \to R_{n-1}^+ \xrightarrow{f_n^+} R'^+$ of any given f^+.*

Proof. (1) Let us first assume that f is *injective*. We note that any extension f^+ is injective [6, V. Cor. 2 to Prop. 1, §2 n.1]. The embedding $K \hookrightarrow K'$ extends to an embedding $K^+ \hookrightarrow K'^+$ (unique up to precomposition by an element of G_K); therefore f extends to an embedding $R^+ \hookrightarrow R'^+$ (unique up to precomposition by an element of G_K).

Let us next assume that f is *surjective*, with kernel $\mathfrak{p}$. Let $\mathfrak{p}^+$ be a prime ideal above $\mathfrak{p}$ in the integral extension R^+. Every monic polynomial P with coefficients in $R^+/\mathfrak{p}^+$, lifts to a monic polynomial with coefficients in R^+, which is a product of linear factors in the absolutely integrally closed domain R^+, hence P is a product of linear factors in $R^+/\mathfrak{p}^+$ as well. We conclude that the fraction field of $R^+/\mathfrak{p}^+$ is algebraically closed. Since $R^+/\mathfrak{p}^+$ is integral over $R' = R/\mathfrak{p}$, it is isomorphic to R'^+, hence f extends to a surjection $R^+ \twoheadrightarrow R'^+$.

The general case is obtained by factoring f into a surjection followed by an embedding.

(2) The case when f is injective has already been treated, and the case when R is normal (*i.e.* $R = R^+ \cap K$) follows from [6, V, Cor. 1 to Prop. 6, §2 n.1].

(3) By (2), for any choice of lifts f_i^+, there exists $\sigma \in G_K$ such that $f_n^+ \circ \cdots f_1^+ \circ \sigma = f^+$. It then suffices to replace f_1^+ by $f_1^+ \circ \sigma$. □

2.1.2. **Remark.** (2) *and* (3) *may fail if the assumption is removed (this is related to the failure of going-down for non-normal rings), see e.g.* [6, V, §2, ex. 13].

2.2. **Perfectoid algebras.** (*cf.* [18], and further [1]; we use a specific perfectoid valuation ring in this paper but the general theory would work over any perfectoid valuation ring). We fix a prime number p. Unless otherwise specified, we denote by $\widehat{\ }$ the p-adic (separated) completion of any ring. We denote by F the Frobenius endomorphism $x \mapsto x^p$ of any ring of characteristic p.

2.2.1. Let k be a perfect field of characteristic p, $W := W(k)$ its Witt ring, $\mathcal{K}^o := W[p^{\frac{1}{p^\infty}}]$, and $\hat{\mathcal{K}}^o$ its completion, which is a *perfectoid valuation ring* (F induces an isomorphism $\hat{\mathcal{K}}^o/p^{\frac{1}{p}} \xrightarrow{\sim} \hat{\mathcal{K}}^o/p$) with residue field k; this is the valuation ring of the perfectoid field $\hat{\mathcal{K}} := \hat{\mathcal{K}}^o[\frac{1}{p}]$.

The *tilt* of $\hat{\mathcal{K}}^o$ is defined to be $\hat{\mathcal{K}}^{\flat o} := \lim_F \hat{\mathcal{K}}^o/p$. This is a perfect complete valuation ring of characteristic p. In fact, denoting by $p^\flat$ the element $(\dots, p^{\frac{1}{p}}, p)$ of $\hat{\mathcal{K}}^{\flat o}$, $\hat{\mathcal{K}}^{\flat o}$ is the $p^\flat$-adic completion of $k[(p^\flat)^{\frac{1}{p^\infty}}]$.

2.2.2. Let B be a p-adically complete and p-torsionfree $\hat{\mathcal{K}}^o$-algebra. Then $B[\frac{1}{p}]$ is canonically a Banach $\mathcal{K}$-algebra, with unit ball $p^{-\frac{1}{p^\infty}}B$. Let $B[\frac{1}{p}]^o \supset B$ be the ring of power-bounded elements of $B[\frac{1}{p}]$. We say that B is *spectral* if the norm of $B[\frac{1}{p}]$ is the spectral seminorm (*i.e.* is power-multiplicative)[2]. This is equivalent to: $p^{-\frac{1}{p^\infty}}B = B[\frac{1}{p}]^o$, or else to: $p^{-\frac{1}{p^\infty}}B$ is integrally closed in $B[\frac{1}{p}]$ [1, 2.3.1 (5)]. This is implied by: B is integrally closed in $B[\frac{1}{p}]$ [8, 8.2.28]. Any spectral algebra is reduced.

2.2.3. We say that B is *perfectoid* if F induces an isomorphism $B/p^{\frac{1}{p}} \xrightarrow{\sim} B/p$.

In that case, for any $\varpi \in \hat{\mathcal{K}}^o$ with $|p|^{\frac{1}{p}} \le |\varpi| < 1$, F induces an isomorphism $B/\varpi \xrightarrow{\sim} B/\varpi^p$. Any perfectoid algebra is spectral.

The p-adic completion of any colimit of perfectoid $\hat{\mathcal{K}}^o$-algebras is perfectoid.

2.2.1. **Proposition.** *Let $\mathfrak{a}$ be an ideal of a perfectoid algebra B generated by a regular sequence $(p, x_2, \dots, x_d)$. Then the $\mathfrak{a}$-adic completion $\hat{B}^{\mathfrak{a}}$ is perfectoid.*

Proof. Since $\mathfrak{a}$ is finitely generated, $\hat{B}^{\mathfrak{a}}$ is $\mathfrak{a}$-adically complete; $(p, x_2, \dots, x_d)$ is a regular sequence in $\hat{B}^{\mathfrak{a}}$, and so is $(x_2^n, \dots, x_d^n, p)$ (*resp.* $(x_2^n, \dots, x_d^n, p^{\frac{1}{p}})$) for every $n > 0$. In particular, $\hat{B}^{\mathfrak{a}}$ is p-torsionfree and p-adically complete. Let us show that the ideal $p\hat{B}^{\mathfrak{a}}$ (*resp.* $p^{\frac{1}{p}}\hat{B}^{\mathfrak{a}}$) is closed for the $\mathfrak{a}$-adic topology, or equivalently, for the $(x_2, \dots, x_d)$-adic topology. Indeed, if an element $b \in \hat{B}^{\mathfrak{a}}$ can be written $pc_n + d_n$ with $d_n \in (x_2^n, \dots, x_d^n)\hat{B}^{\mathfrak{a}}$ for any $n > 0$, then pc_n is a $(x_2, \dots, x_d)$-adic Cauchy sequence. Since $(x_2^n, \dots, x_d^n, p)$ is a regular sequence, c_n is itself a $(x_2, \dots, x_d)$-adic Cauchy sequence, and its limit c satisfies $b = pc$. One sees in the same way that $p^{\frac{1}{p}}\hat{B}^{\mathfrak{a}}$ is $\mathfrak{a}$-adically closed.

It follows that the map $\hat{B}^{\mathfrak{a}}/p \to \widehat{B/p}^{\mathfrak{a}}$ (*resp.* $\hat{B}^{\mathfrak{a}}/p^{\frac{1}{p}} \to \widehat{B/p^{\frac{1}{p}}}^{\mathfrak{a}}$) is an isomorphism. On the other hand, denoting by $\bar{\mathfrak{a}}$ the image of $\mathfrak{a}$ in B/p, one has $\bar{\mathfrak{a}}^{pd} \subset F\bar{\mathfrak{a}} \subset \bar{\mathfrak{a}}^p$, hence $\widehat{B/p}^{\mathfrak{a}} = \widehat{B/p}^{\bar{\mathfrak{a}}} = \widehat{B/p}^{F\bar{\mathfrak{a}}}$. Since B is perfectoid, F induces an isomorphism $\widehat{B/p^{\frac{1}{p}}}^{\mathfrak{a}} = \widehat{B/p^{\frac{1}{p}}}^{\bar{\mathfrak{a}}} \xrightarrow{\sim} \widehat{B/p}^{F\bar{\mathfrak{a}}} = \widehat{B/p}^{\mathfrak{a}}$, hence also an isomorphism $\hat{B}^{\mathfrak{a}}/p^{\frac{1}{p}} \xrightarrow{\sim} \hat{B}^{\mathfrak{a}}/p$. □

2.2.4. The *tilt* of a perfectoid $\hat{\mathcal{K}}^o$-algebra B is $B^\flat := \lim_F B/p$. This is a perfectoid $\hat{\mathcal{K}}^{\flat o}$-algebra, *i.e.* a $p^\flat$-adically complete, $p^\flat$-torsionfree, perfect $\hat{\mathcal{K}}^{\flat o}$-algebra. The natural morphism

$$B^\flat/p^\flat \to B/p \tag{2.1}$$

is an isomorphism. This lifts to a natural isomorphism

$$B^\natural := W(B^\flat)\hat{\otimes}_{W(\hat{\mathcal{K}}^{\flat o})}\hat{\mathcal{K}}^o \xrightarrow{\sim} B \tag{2.2}$$

(this is standard for $p^{-\frac{1}{p^\infty}}B = B[\frac{1}{p}]^o$ (Fontaine-Scholze); it follows that (2.2) is injective in general, and surjectivity is checked by reducing mod. p).

[2] we refer to [1, 2.2] for a discussion of the difference between "spectral" and "uniform" (which means that the norm of $B[\frac{1}{p}]$ is *equivalent* to the spectral seminorm; equivalently, $B[\frac{1}{p}]^o \subset p^{-N}B$ for some positive integer N).

2.2.5. Let R be a complete local domain of mixed characteristic $(0,p)$, fix a compatible system of p^{th}-power roots $p^{\frac{1}{p^i}}$ in R^+, and view R^+ as a $\mathcal{K}^o$-algebra. Then $\widehat{R^+}$ *is perfectoid*. Indeed, $\widehat{R^+}$ is p-torsionfree like R^+; since R^+ is normal, an equation $x^p = py$ in R^+ implies $x/p^{\frac{1}{p}} \in R^+$, hence $\hat{R}^+/p^{\frac{1}{p}} \stackrel{F}{\to} \hat{R}^+/p$ is injective; on the other hand, any element of R^+ admits p-th power roots; one concludes that F induces an isomorphism $\hat{R}^+/p^{\frac{1}{p}} \stackrel{\sim}{\to} \hat{R}^+/p$.

2.3. **Almost perfectoid algebras.** Let B be a p-adically complete and p-torsionfree $\hat{\mathcal{K}}^o$-algebra. Let π be a nonzero element of B which admits a compatible system of p-th power roots $\pi^{\frac{1}{p^i}}$ (which we fix).

2.3.1. We say that B is $\pi^{\frac{1}{p^\infty}}$*-almost perfectoid* if it is *spectral* and if the Frobenius map F induces an almost isomorphism $B/p^{\frac{1}{p}} \stackrel{\sim^a}{\to} B/p$, *i.e.* kernel and cokernel are killed by $\pi^{\frac{1}{p^\infty}}$, cf [1, 3.5.4]. In fact, by spectrality, $B/p^{\frac{1}{p}} \to B/p$ is $p^{\frac{1}{p^\infty}}$-almost injective [1, 3.3.1].

The tilt $B^\flat := \lim_F B/p$ of an almost perfectoid $\hat{\mathcal{K}}^o$-algebra is a perfectoid $\hat{\mathcal{K}}^{\flat o}$-algebra, with a specific element $\pi^\flat = (\ldots, \pi^{\frac{1}{p}}, \pi)$. Moreover $B^\natural := W(B^\flat)\hat{\otimes}_{W(\hat{\mathcal{K}}^{\flat o})}\hat{\mathcal{K}}^o$ is a perfectoid $\hat{\mathcal{K}}^o$-algebra, and the natural morphism $B^\natural \to B$ is a $(p\pi)^{\frac{1}{p^\infty}}$-almost isomorphism (hence a $\pi^{\frac{1}{p^\infty}}$-almost isomorphism if[3] $\pi \in p^{\frac{1}{p^\infty}} B$). Here, in $B^\natural$, $\pi^{\frac{1}{p^i}}$ is identified with the Teichmüller lift $[(\pi^\flat)^{\frac{1}{p^i}}]$. The formation of $B^\natural$ is functorial in B.

By spectrality, $B^\natural[\frac{1}{p}] \to B[\frac{1}{p}]$ is an isometry [1, 3.3.3], and in particular $B^\natural \to B$ is injective.

Obviously, any spectral $\hat{\mathcal{K}}^o$-algebra B which is $\pi^{\frac{1}{p^\infty}}$-almost isomorphic to a $\pi^{\frac{1}{p^\infty}}$-almost perfectoid algebra is $\pi^{\frac{1}{p^\infty}}$-almost perfectoid.

2.3.2. Almost ring theory over the basic setup $(B, \pi^{\frac{1}{p^\infty}} B)$ satisfies the general assumption of [8]: $(\pi^{\frac{1}{p^\infty}} B)^{\otimes 2}$ is flat over B [8, 2.1.7]. However, certain "pathologies" occur when π is a zero-divisor in B. In order to reduce questions about almost perfectoid algebras to the convenient case when π is a non-zero divisor, one may replace B by $\pi^{-\frac{1}{p^\infty}} B$, which is spectral if B is [1, (2.16)]. The elementary but crucial observation is that *in a reduced* $\mathbb{Z}[\pi^{\frac{1}{p^\infty}}]$*-algebra, any* π*-torsion element is* $\pi^{\frac{1}{p^\infty}}$*-torsion*. Therefore $B \to \pi^{-\frac{1}{p^\infty}} B$ *is an almost isomorphism* (so that $\pi^{-\frac{1}{p^\infty}} B$ is almost perfectoid if B is), and *the image of* π *in* $\pi^{-\frac{1}{p^\infty}} B$ *is a non-zero divisor*.

2.4. **Cohen-Macaulay algebras.** Let R be a Noetherian local ring of dimension d, with maximal ideal $\mathfrak{m}$. Let $\underline{x} = (x_1, x_2, \ldots, x_d)$ be a system of parameters, and let B be a (not necessarily Noetherian) R-algebra.

We say that $\underline{x}$ becomes regular in B, or that B is *Cohen-Macaulay with respect to* $\underline{x}$ if for any i, multiplication by x_{i+1} is injective on $B/(x_1, \ldots, x_i)B$, and $B \neq (x_1, \ldots, x_d)B$. It is a *Cohen-Macaulay R-algebra*[4] if it is Cohen-Macaulay with respect to any system of parameters of R.

We shall use freely the following standard facts:

[3]as was noted by a referee (but will not be used in the sequel), this extra condition is actually unnecessary.

[4]this is frequently called a balanced big Cohen-Macaulay algebra in the literature: "balanced big" emphasizes that B is Cohen-Macaulay with respect to *any* system of parameters of R, and is not assumed to be noetherian. See next footnote.

$a)$ if B is Cohen-Macaulay with respect to some $\underline{x}$, its $\mathfrak{m}$-adic completion $\hat{B}^{\mathfrak{m}}$ is a Cohen-Macaulay R-algebra [5, Th. 1.7];
$b)$ a Cohen-Macaulay R-algebra B is also a Cohen-Macaulay S-algebra for any factorization $R \to S \to B$ such that S is local and finite over R (since the image of a system of parameters of R in S is a system of parameters, and B is also $\mathfrak{m}_S$-adically complete);
$c)$ an algebra over a regular local ring R is Cohen-Macaulay if and only it is faithfully flat [13, 2.1.d];
$d)$ if R is a complete local domain, any Cohen-Macaulay R-algebra B with respect to some $\underline{x}$ is faithful (by Cohen's structure theorem, the domain R is finite over some complete regular local subdomain A, and one may apply items $a)$ and $c)$ to the composed map $A \to R \to B \to \hat{B}^{\mathfrak{m}}$; alternatively, one can invoke [11, Cor. 10.6]).

2.5. **Almost Cohen-Macaulay algebras.** Let B be an R-algebra, and let $\pi^{\frac{1}{p^\infty}}$ be a compatible system of p^{th}-power roots of some nonzero element $\pi \in B$.

We say that $\underline{x}$ becomes $\pi^{\frac{1}{p^\infty}}$-almost regular in B, or that B is $\pi^{\frac{1}{p^\infty}}$*-almost Cohen-Macaulay with respect to* $\underline{x}$ if for any i, multiplication by x_{i+1} is almost injective on $B/(x_1, \dots, x_i)B$, and $B/(x_1, \dots, x_d)B$ is not almost zero, *i.e.* $\pi^{\frac{1}{p^\infty}}B \not\subset (x_1, \dots, x_d)B$. Since $(x_1, \dots, x_d)R$ contains some power of $\mathfrak{m}$, the last condition is equivalent to $\pi^{\frac{1}{p^\infty}}B \not\subset \mathfrak{m}B$, *cf.* [2, 4.1].

This terminology may be slightly misleading since "Cohen-Macaulay" does not formally imply "almost Cohen-Macaulay": $\pi^{\frac{1}{p^\infty}}B \not\subset \mathfrak{m}B$ is stronger than $B \neq \mathfrak{m}B$. The following proposition allows us to settle this issue in some generality.

2.5.1. **Proposition.** *Let R be a complete Noetherian local domain with maximal ideal $\mathfrak{m}$, and let B be a Cohen-Macaulay algebra with respect to some system of parameters $\underline{x}$ of R. Assume that B contains a system of p^{th}-power roots $\pi^{\frac{1}{p^\infty}}$ of some nonzero element $\pi \in B$.*

(1) *If B is $\mathfrak{m}$-adically complete and $\pi^{\frac{1}{p^\infty}}B \cap R \neq 0$, then B is a $\pi^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay R-algebra.*
(2) *Let $\mathfrak{p}$ be a prime ideal of R such that $\pi^{\frac{1}{p^\infty}}B \cap R \not\subset \mathfrak{p}R$. Then $B/(\mathfrak{m}B + \sqrt{\mathfrak{p}B})$ is not $\pi^{\frac{1}{p^\infty}}$-almost zero.*

Proof. (1) (not used in the sequel - *cf.* also [2, 1.1.2]) We have to show that $\pi^{\frac{1}{p^\infty}}B \not\subset \mathfrak{m}B$. By Cohen's structure theorem, the domain R is finite over some complete regular local subdomain A. The assumption $\pi^{\frac{1}{p^\infty}}B \cap R \neq 0$ then implies $\pi^{\frac{1}{p^\infty}}B \cap A \neq 0$ (using a norm argument). On the other hand, since B is $\mathfrak{m}$-adically complete, it is a Cohen-Macaulay R-algebra, hence faithfully flat over A. In particular $\mathfrak{m}_A^n B \cap A = \mathfrak{m}_A^n$ for every n. Their intersection is 0 by Krull. Since $\mathfrak{m}_A R$ is $\mathfrak{m}$-primary, $\cap\, \mathfrak{m}^n B \cap A = 0$. If the idempotent ideal $\pi^{\frac{1}{p^\infty}}B$ is contained in $\mathfrak{m}B$, it is contained in $\cap\, \mathfrak{m}^n B$, which contradicts $\pi^{\frac{1}{p^\infty}}B \cap A \neq 0$.

(2) Let us set $R' = R/\mathfrak{p}$, a complete local domain with maximal ideal $\mathfrak{m}' = \mathfrak{m}/\mathfrak{p}$. We cannot apply (1) to the R'-algebra $B/\sqrt{\mathfrak{p}B}$, which might not be Cohen-Macaulay; instead, following O. Gabber's suggestion, we will use Hochster's solid closure theory [11].

Let us write $\sqrt{\mathfrak{p}B}$ as a filtered union of finitely generated ideals I_α of B containing $\mathfrak{p}B$. Since B is a Cohen-Macaulay algebra with respect to $\underline{x}$ and $(\underline{x})R$ is $\mathfrak{m}$-primary, B has a nonzero R-dual [11, 2.4]. It follows that $B/\mathfrak{p}B$ has a nonzero R'-dual [11, 2.12], and further that every B/I_α has a nonzero R'-dual [11, 2.1 (m)]. By [11, 5.10], it follows that for any given ideal J of R' and every α, the intersection $J(B/I_\alpha) \cap R'$ in B/I_α (which equals the intersection $(JB + I_\alpha) \cap R'$ in B) is contained in the integral closure $\bar{J}$ of J (in the terminology of [11], $J(B/I_\alpha) \cap R'$ belongs to the solid closure of J). Their filtered union $J(B/\sqrt{\mathfrak{p}B}) \cap R'$ is therefore contained in $\bar{J}$.

Now $B/(\mathfrak{m}B + \sqrt{\mathfrak{p}B}) = (B/\sqrt{\mathfrak{p}B})/\mathfrak{m}'(B/\sqrt{\mathfrak{p}B})$, and by the same argument as in (1), it suffices to see that $\cap_n \mathfrak{m}'^n(B/\sqrt{\mathfrak{p}B}) \cap R' = 0$. Applying the above observation to $J = \mathfrak{m}'^n$, this is derived from the classical vanishing of $\cap \overline{\mathfrak{m}'^n}$ [15, 5.3.4]. $\square$

2.5.2. **Remark.** *In 4.3 and 4.4, Proposition 2.5.1 (2) will ensure that our constructions do not lead to the (almost) zero algebra. On the other hand, B will always be reduced, so that the condition $\pi \neq 0$ is equivalent to the non-triviality of the basic setup, i.e. $\pi^{\frac{1}{p^\infty}} B \neq 0$, and also to $\pi^{-\frac{1}{p^\infty}} B \neq 0$.*

3. Getting rid of "almost": Shimomoto's construction.

3.1. Let R be a complete Noetherian local domain of mixed characteristic with algebraically closed residue field $k = k^+$ of characteristic p. Let $(x_1, x_2, \ldots, x_d)$ be a system of parameters for R, with $x_1 = p$.

The following synthetizes, strengthens, and recasts the results of [20][5] (and their proof) according to our needs.

3.1.1. **Theorem.** *Let B be a $\pi^{\frac{1}{p^\infty}}$-almost $\hat{\mathcal{K}}^o$-perfectoid, almost Cohen-Macaulay R-algebra with respect to $\underline{x}$ for some nonzero $\pi \in p^{\frac{1}{p^\infty}} B$ (it is tacitly assumed that the $W(k)$-structures coming from R and $\hat{\mathcal{K}}^o$ coincide).*

1. *There exists a $\hat{\mathcal{K}}^o$-perfectoid Cohen-Macaulay R-algebra C (which one may assume to be $\mathfrak{m}_R$-adically complete) and a morphism $B^\natural \to C$.*
2. *One may assume moreover that C is an R^+-algebra.*

The assumption $\pi \in p^{\frac{1}{p^\infty}} B$ ensures that the $\pi^{\frac{1}{p^\infty}}$ and $(p\pi)^{\frac{1}{p^\infty}}$ basic setups coincide.

The existence of $B^\natural \to C$ will be crucial in the proof of Theorem 1.2.1 (caution: $B^\natural$ may not be an R-algebra). In its standard version, Hochster's modification process to construct a Cohen-Macaulay R-algebra C, as applied for instance in [2], starts with an almost Cohen-Macaulay B as an auxiliary object, without relating B and C explicitly (the almost Cohen-Macaulay condition is there to ensure that the process does not degenerate). Dietz's work remedies this shortfall by providing a map $B \to C$ in char. p, while Shimomoto "transposes" it into mixed characteristic by tilting[6].

For convenience, we restate the fragment of Dietz's theory that we need as follows:

3.1.2. **Proposition.** *Let S be a complete Noetherian local k-algebra with residue field k.*

1. *Any $(\pi^\flat)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay S-algebra D maps to a $\mathfrak{m}_S$-adically separated, Cohen-Macaulay S-algebra which is a perfect domain (resp. $\hat{\mathcal{K}}^{\flat o}$-perfectoid, if $p^\flat$ is a parameter of S).*
2. *Let (D_α) be a directed system of $(\pi^\flat_\alpha)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay S-algebras (for some $\pi^\flat_\alpha \in D_\alpha \setminus 0$). Then $\operatorname{colim} D_\alpha$ maps to a Cohen-Macaulay S-algebra which is a perfect domain (resp. $\hat{\mathcal{K}}^{\flat o}$-perfectoid, if $p^\flat$ is a parameter of S).*

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[5] actually in [20], the $\mathfrak{m}_R$-adic completion is not performed, and one only gets a Cohen-Macaulay algebra C with respect to $(p, x_2, \ldots, x_d)$. The terminological variability in the literature (big CM, balanced big CM...) may be a source of misunderstanding/overinterpretation.

[6] after the release of the first version of this paper, I learned that Gabber has an alternative way of passing from almost perfectoid, almost Cohen-Macaulay algebras to perfectoid Cohen-Macaulay algebras, using an ultraproduct technique.

Proof. (1) In the terminology of [7], the fact that D is $(\pi^\flat)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay translates into: $\pi^\flat$ is a "durable colon-killer" over S. Thus D is a "seed" by [7, 4.8], and any seed maps to a Cohen-Macaulay algebra which is perfect and $\mathfrak{m}_S$-adically separated by [7, 3.7] and is also a domain by [7, 7.8]. Hence after $p^\flat$-adic completion, D maps to a $\hat{\mathcal{K}}^{\flat o}$-perfectoid Cohen-Macaulay S-algebra if $p^\flat$ is a parameter of S.

For (2), one observes that a directed colimit of seeds is a seed [7, 3.2]. □

Proof. (of Theorem 3.1.1).

(1) $a)$ We may replace B by any B-algebra which is $\pi^{\frac{1}{p^\infty}}$-almost $\hat{\mathcal{K}}^o$-perfectoid and almost Cohen-Macaulay R-algebra with respect to $\underline{x}$. For instance,

$i)$ we may replace B by the $\pi^{\frac{1}{p^\infty}}$-almost isomorphic algebra $\pi^{-\frac{1}{p^\infty}}B$, so that π *is not a zero divisor in* B and $B = \pi^{-\frac{1}{p^\infty}}B^\natural$ (since $B^\natural$ is $\pi^{\frac{1}{p^\infty}}$-almost isomorphic to B);

$ii)$ we may then replace B by $B[\frac{1}{p}]\langle \underline{x}^{\frac{1}{p^\infty}}\rangle^o$; in this situation, B *contains a system of* p^{th}*-power roots* $x_i^{\frac{1}{p^\infty}}$ *of each* x_i. In order to see that $B^{\cdot} := B[\frac{1}{p}]\langle \underline{x}^{\frac{1}{p^\infty}}\rangle^o$ is indeed $\pi^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay with respect to $\underline{x}$, it suffices to see that $B^{\cdot}/p$ is $\pi^{\frac{1}{p^\infty}}$-almost faithfully flat over B/p. We note that $B^{\cdot}$ is $p^{\frac{1}{p^\infty}}$-almost isomorphic to $\widehat{\operatorname{colim}}\, B_i$, where B_i is the integral closure of $B[\underline{x}^{\frac{1}{p^i}}]$ in $B[\underline{x}^{\frac{1}{p^i}}, \frac{1}{p}]$. We also note that B_i is $\pi^{\frac{1}{p^\infty}}$-almost isomorphic to the colimit (union) on j of the integral closure B_{ij} of $B[(\pi^{\frac{1}{p^j}}\underline{x})^{\frac{1}{p^i}}]$ in $B[(\pi^{\frac{1}{p^j}}\underline{x})^{\frac{1}{p^i}}, \frac{1}{p}]$. Since $B^\natural \to B$ is a $\pi^{\frac{1}{p^\infty}}$-almost isomorphism, $\pi^{\frac{1}{p^j}}\underline{x}$ is a d-uple in $B^\natural$, and B_{ij} is $\pi^{\frac{1}{p^\infty}}$-almost isomorphic to the integral closure $B_{ij}^\natural$ of $B^\natural[(\pi^{\frac{1}{p^j}}\underline{x})^{\frac{1}{p^i}}]$ in $B^\natural[(\pi^{\frac{1}{p^j}}\underline{x})^{\frac{1}{p^i}}, \frac{1}{p}]$.

Applying [2, 2.5.2] to the perfectoid algebra $B^\natural$, we see that for any j, $(\widehat{\operatorname{colim}_i}\, B_{ij}^\natural)/p$ is $p^{\frac{1}{p^\infty}}$-almost faithfully flat over $B^\natural/p$. Passing to the colimit on j, we conclude that $B^{\cdot}/p$ is $\pi^{\frac{1}{p^\infty}}$-almost faithfully flat over B/p.

$b)$ These reductions being done (namely: $i)$, $ii)$, and $i)$ again), let us first assume that R *is normal*. Let $\tilde{R} := R[\frac{1}{p}]^{et,o}$ be the integral closure of R in the maximal etale extension of $R[\frac{1}{p}]$ in some fixed algebraic closure K^+ of the field of fractions of R. Let us write $\tilde{R} = \operatorname{colim} R_\alpha$ as a directed colimit of finite normal extensions of R such that $R_\alpha[\frac{1}{p}]$ is etale over $R[\frac{1}{p}]$. The p-adic completion $\hat{\tilde{R}} := \widehat{\operatorname{colim}}\, R_\alpha$ of $\tilde{R}$ is $\hat{\mathcal{K}}^o$-perfectoid, *cf. e.g.* [19, 10.1].

Let $\tilde{B}$ be the integral closure of B in $\tilde{R} \otimes_R B[\frac{1}{\pi}]$. We next show that its completion $\hat{\tilde{B}}$ *is a* $\pi^{\frac{1}{p^\infty}}$*-almost perfectoid* $\hat{\tilde{R}}$*-algebra, and* $\hat{\tilde{B}}/p$ *is* $\pi^{\frac{1}{p^\infty}}$*-almost faithfully flat over* B/p.

Indeed, $\hat{\tilde{B}}$ is the completed colimit of the integral closures $B_{(\alpha)}$ of $B = \pi^{-\frac{1}{p^\infty}}B^\natural$ in $R_\alpha \otimes_R B[\frac{1}{\pi}]$. Since $R_\alpha \otimes_R B[\frac{1}{\pi}]$ is finite etale over $B[\frac{1}{\pi}] = B^\natural[\frac{1}{\pi}]$, the perfectoid Abhyankar lemma [1, 0.3.1, 5.3.1][7] applies to $(B^\natural[\frac{1}{p}],\ R_\alpha \otimes_R B[\frac{1}{\pi}])$ and shows that $(B_{(\alpha)})^\natural$ is perfectoid, $B_{(\alpha)}$ is $\pi^{\frac{1}{p^\infty}}$-almost perfectoid, and $B_{(\alpha)}/p$ is $\pi^{\frac{1}{p^\infty}}$-almost faithfully flat over B/p.

[7] see also a very short account of its proof in [3]. In the notation of [1], $B^\natural[\frac{1}{p}]$ is $\mathcal{A}$, $R_\alpha \otimes_R B[\frac{1}{\pi}]$ is $\mathcal{B}'$ and $B_{(\alpha)}$ is $\tilde{\mathcal{B}}^o$.

Since $\underline{x}$ is a $\pi^{\frac{1}{p^\infty}}$-almost regular sequence in B, we deduce that the image of $(x_2,\dots,x_d)$ is also a $\pi^{\frac{1}{p^\infty}}$-almost regular sequence in $\hat{\tilde{B}}/p$. Tilting $\hat{\tilde{B}}$, we get a perfectoid $\hat{\mathcal{K}}^{\flat o}$-algebra $\hat{\tilde{B}}^\flat$ and a $\pi^{\flat\frac{1}{p^\infty}}$-almost regular sequence $\underline{x}^\flat = (p^\flat, x_2^\flat,\dots,x_d^\flat)$, where $x_i^\flat = (\dots, x_i^{\frac{1}{p}}, x_i)$ (note that the $\pi^{\flat\frac{1}{p^\infty}}$-almost isomorphism $\hat{\tilde{B}}^\flat/p^\flat \to \hat{\tilde{B}}/p$ sends $\underline{x}^\flat$ to $\underline{x}$).

By Proposition 3.1.2 (1), one can map $\hat{\tilde{B}}^\flat$ to a $\hat{\mathcal{K}}^{\flat o}$-perfect(oid) Cohen-Macaulay $k[[p^\flat, x_2^\flat,\dots,x_d^\flat]]$-algebra. Untilting, we get a $\hat{\mathcal{K}}^o$-perfectoid Cohen-Macaulay algebra with respect to $\underline{x}$. Completing $(p,x_2,\dots,x_d)$-adically, we get a $\hat{\mathcal{K}}^o$-perfectoid Cohen-Macaulay R-algebra C (Proposition 2.2.1), as well as morphisms $B^\natural \to \hat{\tilde{B}}^\natural \to C$.

$c)$ We next drop the assumption that R is normal. Let R^{n} be the normalization of R (which is again a complete Noetherian local domain), and $g \in R\setminus\{0\}$ be such that $R^{\mathrm{n}}[\frac{1}{pg}] = R[\frac{1}{pg}]$. We may (by the same argument as above) assume that g is a non-zero divisor in B and that B contains $g^{\frac{1}{p^\infty}}$. Let $\tilde{B}^{\mathrm{n}}$ be the integral closure of B in $(\tilde{R}^{\mathrm{n}}\otimes_R B)[\frac{1}{\pi g}]$. We then construct C and morphisms $B^\natural \to \hat{\tilde{B}}^{\mathrm{n}\natural} \to C$ as above, on replacing $(\tilde{R},\tilde{B},\pi)$ by $(\tilde{R}^{\mathrm{n}},\tilde{B}^{\mathrm{n}},\pi g)$. This finishes the proof of item (1).

3.1.3. **Remark.** *Since $(\tilde{R}^{\mathrm{n}}\otimes_R B)[\frac{1}{\pi g}] = (\tilde{R}^{\mathrm{n}}\otimes_R B)[\frac{1}{\pi g}][\frac{1}{p}]$, $\tilde{B}^{\mathrm{n}}$ is integrally closed in $\tilde{B}^{\mathrm{n}}[\frac{1}{p}]$, hence spectral, so that $\tilde{B}^{\mathrm{n}\natural} \to \tilde{B}^{\mathrm{n}}$ is injective (cf. 2.3.1).*

(2) Let us write R^+ as a directed colimit of finite normal R-subalgebras R_β, which are complete local domains with residue field $k = k^+$ ("local" because R is Henselian and R_β is a domain). Setting $\tilde{R}_\beta := R_\beta[\frac{1}{p}]^{et,o}$ as above, we also have $R^+ = \operatorname{colim} \tilde{R}_\beta$. Recall that once a compatible system of roots $p^{\frac{1}{p^i}}$ is chosen in R^+, the p-adic completion $\widehat{R^+}$ is perfectoid over $\hat{\mathcal{K}}^o$ (2.2.5).

Let $\underline{g}_\beta = (g_{\beta,1}, g_{\beta,2},\dots)$ be a finite sequence of elements of $R_\beta\setminus\{0\}$ such that $R_\beta[\frac{1}{p\Pi g_{\beta,i}}]$ is etale over $R[\frac{1}{p\Pi g_{\beta,i}}]$. We consider the directed system formed by pairs $\underline{\beta} = (\beta, \underline{g}_\beta)$ (where the order $(\beta,\underline{g}_\beta) \le (\beta',\underline{g}_{\beta'})$ if $\beta\le\beta'$ and $\underline{g}_\beta$ is part of the sequence $\underline{g}_{\beta'}$). We define $B_{\underline{\beta}} := (\Pi g_{\beta,i})^{-\frac{1}{p^\infty}} B\langle \underline{g}_\beta^{\frac{1}{p^\infty}}\rangle[\frac{1}{p}]^o$, which is $(\pi\underline{g}_\beta)^{\frac{1}{p^\infty}}$-almost perfectoid. When $\underline{\beta}$ varies, they form a directed system of R-algebras. By the same argument as in $(1)\, ii)$ above (involving [2, 2.5.2]), $B_{\underline{\beta}}/p$ is $(\pi\underline{g}_\beta)^{\frac{1}{p^\infty}}$-almost faithfully flat over B/p.

We further define $\tilde{B}_{\underline{\beta}}$ to be the integral closure of $B_{\underline{\beta}}\otimes_R \tilde{R}_\beta$ in $(B_{\underline{\beta}}\otimes_R \tilde{R}_\beta)[\frac{1}{p\Pi g_{\beta,i}}]$. They form a directed system of $\tilde{R}_\beta$-algebras. Since $B_{\underline{\beta}}$ contains a system of p-th power roots $(\Pi g_{\beta,i})^{\frac{1}{p^\infty}}$ of the non-zero divisor $\Pi g_{\beta,i} \in B_{\underline{\beta}}$, the perfectoid Abhyankar lemma shows (as above) that $\hat{\tilde{B}}_{\underline{\beta}}$ is $(\pi\underline{g}_\beta)^{\frac{1}{p^\infty}}$-almost perfectoid, and that $(\tilde{B}_{\underline{\beta}})/p$ is $(\pi\underline{g}_\beta)^{\frac{1}{p^\infty}}$-almost faithfully flat over $B_{\underline{\beta}}/p$, hence over B/p, so that $\hat{\tilde{B}}_{\underline{\beta}}$ is $(\pi\underline{g}_\beta)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay with respect to $\underline{\mathrm{x}}$. We may then imitate the argument in (1): the $\hat{\tilde{R}}_\beta^\flat$-algebras $\hat{\tilde{B}}_{\underline{\beta}}^\flat$ form a directed system of $(\pi^\flat \underline{g}_\beta^\flat)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay $k[[\underline{x}^\flat]]$-algebras, hence their $p^\flat$-adically completed colimit $\widehat{\operatorname{colim}_{\underline{\beta}}}\, \hat{\tilde{B}}_{\underline{\beta}}^\flat \cong (\widehat{\operatorname{colim}_{\underline{\beta}}}\, \hat{\tilde{B}}_{\underline{\beta}}^\natural)^\flat$ maps to a $\hat{\mathcal{K}}^{\flat o}$-perfect(oid) Cohen-Macaulay $k[[p^\flat, x_2^\flat,\dots,x_d^\flat]]$-algebra by Proposition 3.1.2 (2). Untilting and completing $(p,x_2,\dots,x_d)$-adically, we get the perfectoid Cohen-Macaulay

R-algebra C (which is an algebra over $\widehat{R^+} = \widehat{\mathrm{colim}}\, \hat{\hat{R}}_\beta$), and we also get a morphism $B^\natural \to \widehat{\mathrm{colim}}\, \hat{\hat{B}}^\natural_{\underline{\beta}} \to C$. $\square$

3.2. In [2, §4], the existence of a Cohen-Macaulay R-algebra is shown by first constructing a $\pi^{\frac{1}{p^\infty}}$-almost perfectoid $\pi^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay R-algebra B (where $\pi \in pR$, $R[\frac{1}{\pi}]$ is etale over $W(k)[[x_2,\dots,x_d]][\frac{1}{\pi}]$, and B is π-torsionfree), and then applying the technique of algebra modifications to B. Using Theorem 3.1.1 instead of the latter (after reducing to the case of an algebraically closed residue field, *cf.* 4.1.1 below), one gets the following more precise result:

3.2.1. **Theorem.** [20][8] *For any complete Noetherian local domain R of mixed characteristic with perfect residue field k, there exists a $\hat{\mathcal{K}}^o$-perfectoid $\mathfrak{m}_R$-adically complete Cohen-Macaulay R-algebra C. Moreover, one can assume that C is an R^+-algebra (in such a way that the $\mathcal{K}^o$-structures coming from R^+ and $\hat{\mathcal{K}}^o$ coincide).* $\square$

4. Proof of Theorem 1.2.1.

4.1. Reduction of Theorem 1.2.1 to the case of a surjection, and a stronger statement in that case.

4.1.1. First, we can reduce to the case when k' is algebraically closed. Indeed, it suffices to replace R' by an extension R'' defined as follows:

i) if R' is of mixed characteristic, let Λ be a coefficient ring and $(x'_1 = p, x'_2, \dots, x'_{d'})$ be a system of parameters; then R' is a finite extension of $\Lambda[[x'_2,\dots,x'_{d'}]]$ by Cohen's theorem, and it follows that $R'\hat{\otimes}_\Lambda W(k'^+) \cong R' \otimes_{\Lambda[[x'_2,\dots,x'_{d'}]]} W(k'^+)[[x'_2,\dots,x'_{d'}]]$ is a complete Noetherian local R'-algebra of dimension d'; we take for R'' the quotient of $R'\hat{\otimes}_\Lambda W(k'^+)$ by some minimal prime with $\dim R'' = d'$;

ii) if R' is of characteristic p, we make a similar construction replacing $\Lambda[[x'_2,\dots,x'_{d'}]]$ by $k[[x'_1, x'_2, \dots, x'_{d'}]]$.

According to [4, 1.1], one has a "Cohen factorization" $R \to R^\cdot \to R' = R^\cdot/\mathfrak{p}$ where $R \to R^\cdot$ is flat. Since every system of parameters for R extends to a system of parameters for $R^\cdot$ by flatness, any Cohen-Macaulay $R^\cdot$-algebra is also a Cohen-Macaulay R-algebra. It thus suffices to treat the case of $R^\cdot \to R'$; in other terms, we may assume from the beginning that $R \to R'$ *is surjective* - and in particular R and R' *have the same (algebraically closed) residue field* $k = k'$.

These reductions are compatible with the prescription of f^+ in Theorem 1.2.1, since we may replace at once R by its normalization, R' by the compositum of R' and the image of the normalization of R in R'^+, and apply Proposition 2.1.1 (3) to the sequence $R \to R^\cdot \to R'$.

4.1.2. This discussion shows that Theorem 1.2.1 follows from the following one (in which we also include the case of char. p for convenience):

4.1.1. **Theorem.** *Let R be a complete Noetherian local domain with algebraically closed residue field k of char. $p > 0$, and let C be an R^+-algebra which is an $\mathfrak{m}_R$-adically complete Cohen-Macaulay R-algebra. If R is of mixed characteristic, we assume moreover that C is $\hat{\mathcal{K}}^o$-perfectoid (in such a way that the $\mathcal{K}^o$-structures coming from R^+ and $\hat{\mathcal{K}}^o$ coincide - such a C exists by Theorem 3.2.1).*

[8] see footnote 5.

Let $f : R \to R' := R/\mathfrak{p}$ be the quotient map by a prime ideal, and let $f^+ : R^+ \to R'^+$ be a lift of f.

Then there exists a R'^+-algebra C' which is an $\mathfrak{m}_{R'}$-adically complete Cohen-Macaulay R'-algebra, $\hat{\mathcal{K}}^o$-perfectoid if R' is of mixed characteristic (resp. perfect if R' is of characteristic p), and which fits into the commutative diagram

(4.1)
$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ \downarrow & & \downarrow \\ R^+ & \xrightarrow{f^+} & R'^+ \\ \downarrow & & \downarrow \\ C & \longrightarrow & C'. \end{array}$$

The fact that C is given "in advance" is crucial, as it permits us to compose such diagrams for a chain of quotients $R \to R/\mathfrak{p}_1 \to \cdots \to R/\mathfrak{p}_n$, and conversely to decompose $R \to R'$ into successive quotients by primes of height 1 (taking into account Proposition 2.1.1).

We thus may and shall assume that $\mathfrak{p}$ *is of height* 1. Replacing R by its normalization (and $\mathfrak{p}$ by a prime over it), we may and shall assume in addition that R *is normal*.

4.2. **Proof of Theorem 4.1.1: the case when R is of characteristic p.** This is a straightforward consequence of Dietz's results. We write R^+ as a directed colimit of finite normal sub-R-algebras R_β, and note that R_β is also a complete local domain with residue field k. Then $R'_\beta = f^+(R_\beta)$ is a complete local domain, and f^+ is the colimit of the induced maps $R_\beta \twoheadrightarrow R'_\beta$.

According to [7, 8.9], $C \otimes_{R_\beta} R'_\beta$ is a seed, and so is $C \otimes_{R^+} R'^+ = \operatorname{colim}\,(C \otimes_{R_\beta} R'_\beta)$ by [7, 3.2]. By [7, 3.7], $C \otimes_{R^+} R'^+$ thus maps to a perfect Cohen-Macaulay R'-algebra, and this provides the desired morphism $C \to C'$, which is compatible with f^+. $\square$

4.3. **Proof of Theorem 4.1.1: the case when R' is of mixed characteristic.** In this case, $R_\mathfrak{p}$ is a discrete valuation ring of equal characteristic 0. We proceed in five steps.

4.3.1. *Choosing an appropriate system of parameters of R.*

We take $x_1 = p$. We choose $x_2 \in \mathfrak{p}$ such that x_2 generates $\mathfrak{p}R_\mathfrak{p}$ and does not lie in any minimal prime ideal $\mathfrak{q}$ of R above p.

Such an element exists. Indeed, since $R/\mathfrak{p}$ is of mixed characteristic, $\mathfrak{p}$ is not contained in any (height one prime) $\mathfrak{q}$. By prime avoidance there exists $y \in \mathfrak{p}$ such that y does not belong to any $\mathfrak{q}$. If y generates $\mathfrak{p}R_\mathfrak{p}$, we take $x_2 = y$. Otherwise, let $x \in \mathfrak{p}$ generate $\mathfrak{p}R_\mathfrak{p}$; by the affine version of prime avoidance[9], there exists $x_2 \in x + yR$ which does not belong to any $\mathfrak{q}$.

The pair (p, x_2) is then part of a system of parameters $\underline{x} = (x_1, x_2, \ldots, x_d)$ of R, which we fix. We also fix a compatible sequence of roots[10] $x_2^{\frac{1}{e}} \subset R^+$ ($e \in \mathbb{N}_{>0}$), and still denote

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[9]due to E. Davis, *cf.* [17, th. 124] (I am indebted to P. Roberts for this reference). We recall the proof for convenience. Let us assume on the contrary that $x + yR \subset \cup_1^n \mathfrak{q}_i$ (with n minimal), and select m such that $x \in \mathfrak{q}_i, i \le m \le n$ but $x \notin \cup_{m+1}^n \mathfrak{q}_i$. For every $j > m$, let $z_j \in \mathfrak{q}_j$ be such that $z_j \notin \mathfrak{q}_\ell$, $\ell \neq j$. If $Ry \subset \cup_1^m \mathfrak{q}_i$, then $(x, y)R \subset \mathfrak{p}_i$ for some $i \le m$ by prime avoidance: a contradiction. Therefore there exists $y' \in Ry$ not in $\cup_1^m \mathfrak{q}_i$, and then $x_2 = x + y' \Pi_{m+1}^n z_j \notin \cup_1^n \mathfrak{q}_i$.

Actually, since C is reduced, it is not essential for the sequel to assume that x_2 generates $\mathfrak{p}R_\mathfrak{p}$.

[10]In perfectoid contexts, it is usual to extract p-power roots, not all roots. Our choice to extract all roots of x_2 is related to our use of Abhyankar's lemma in 4.3.5 below.

by the same symbols their images in C. We denote by $(x_2^{\frac{1}{\infty}})$ the ideal $\cup x_2^{\frac{1}{e}} C$ and by $(x_2^{\frac{1}{\infty}})^-$ its closure.

4.3.2. *$\bar{C} := C/(x_2^{\frac{1}{\infty}})^-$ is perfectoid, and $\underline{x}' := (p, x_3, \dots, x_d)$ becomes a regular sequence in $\bar{C}$.*

For any positive integer e, $R[x_2^{\frac{1}{e}}]$ is a finite extension of the regular ring $W(k)[[x_2^{\frac{1}{e}}, x_3, \dots, x_d]]$. On the other hand, C is a Cohen-Macaulay $R[x_2^{\frac{1}{e}}]$-algebra, hence faithfully flat over $W(k)[[x_2^{\frac{1}{e}}, x_3, \dots, x_d]]$, hence also over their colimit $W(k)[[x_2^{\frac{1}{\infty}}, x_3, \dots, x_d]]$. Thus $C/(x_2^{\frac{1}{\infty}})$ is faithfully flat on $A' := W(k)[[x_3, \dots, x_d]]$, and so is its (separated) completion $\bar{C}$ [2, 1.1.1]. Therefore $\bar{C}$ is a Cohen-Macaulay A'-algebra (hence p-torsionfree).

In turn, C being $\hat{\mathcal{K}}^o$-perfectoid, Frobenius induces isomorphisms

$$(x_2^{\frac{1}{pe}} C)/p^{\frac{1}{p}} = x_2^{\frac{1}{pe}} (C/p^{\frac{1}{p}}) \xrightarrow{\sim} x_2^{\frac{1}{e}} (C/p) = (x_2^{\frac{1}{e}} C)/p. \tag{4.2}$$

Passing to the colimit on e (ordered by divisibility), we conclude that $\bar{C}$ is $\hat{\mathcal{K}}^o$-perfectoid.

4.3.3. *Constructing an almost perfectoid, almost Cohen-Macaulay R'-algebra B' (with respect to $\underline{x}'$).* Since $R_{\mathfrak{p}}$ is a discrete valuation ring with uniformizer x_2, there exists $\pi \in R \setminus \mathfrak{p}$ such that

$$\pi \mathfrak{p} \subset x_2 R. \tag{4.3}$$

Since $R' = R/\mathfrak{p}$ is p-torsionfree, we may also assume that p divides π.

We fix a compatible system of p^{th}-power roots $\pi^{\frac{1}{p^i}}$ in R^+ and view C and $\bar{C}$ as $\mathbb{Z}[\pi^{\frac{1}{p^\infty}}]$-algebras. Since $\bar{C}$ is perfectoid, hence reduced, by (4.3) one has the inclusions

$$\pi^{\frac{1}{p^\infty}} \sqrt{\mathfrak{p} C} \subset \sqrt{x_2 C} \subset (x_2^{\frac{1}{\infty}})^-. \tag{4.4}$$

In particular $\bar{C}/p = C/(pC + x_2^{\frac{1}{\infty}} C)$ is $\pi^{\frac{1}{p^\infty}}$-almost isomorphic to $C/(pC + \sqrt{\mathfrak{p}C})$. Since $\pi \in R \setminus \mathfrak{p}$, Proposition 2.5.1 (2) shows that $C/(\mathfrak{m}C + \sqrt{\mathfrak{p}C})$ is not almost zero, and we conclude that $\bar{C}$ is a $\pi^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay R/x_2-algebra with respect to $\underline{x}'$.

However, $\bar{C}$ may have $\pi^{\frac{1}{p^\infty}}$-torsion, hence not be a R'-algebra. We introduce

$$B' := \pi^{-\frac{1}{p^\infty}} \bar{C},$$

a sub-$\mathcal{K}^o$-algebra of $\bar{C}[\frac{1}{\pi}]$. It is $\pi^{\frac{1}{p^\infty}}$-almost perfectoid and $\pi^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay with respect to $\underline{x}'$ since $\bar{C} \to B'$ is a $\pi^{\frac{1}{p^\infty}}$-almost isomorphism, *cf.* 2.3.2. By (4.3), $\mathfrak{p}$ goes to π-torsion in B' (hence to 0), so that $R \to B'$ factors through R'.

4.3.4. *Obtaining C' from Shimomoto's construction.* We may apply Theorem 3.1.1 to B', which is a $\pi^{\frac{1}{p^\infty}}$-almost $\hat{\mathcal{K}}^o$-perfectoid, almost Cohen-Macaulay R'-algebra with respect to $\underline{x}'$. This provides a $\hat{\mathcal{K}}^o$-perfectoid $\mathfrak{m}_{R'}$-adically complete Cohen-Macaulay R'-algebra C' (and we may assume that C' is an R'^+-algebra), together with morphisms of perfectoid algebras $B'^\natural \to \hat{\hat{B}}'^{\mathrm{n}\natural} \to C'$. Moreover, $\hat{\hat{B}}'^{\mathrm{n}\natural}$ is an $\hat{\hat{R}}'^{\mathrm{n}}$-algebra. Since $\bar{C}$ is perfectoid, the composed R-morphism $\bar{C} \to B' \to \hat{\hat{B}}'^{\mathrm{n}}$ factors through $\hat{\hat{B}}'^{\mathrm{n}\natural}$. The diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & R' \\ \downarrow & & \downarrow \\ \bar{C} & \longrightarrow & \hat{\hat{B}}'^{\mathrm{n}\natural} \end{array} \tag{4.5}$$

commutes, because it commutes after composition with the *injective* map $\hat{\tilde{B}}'^{\mathrm{n}\natural} \to \hat{\tilde{B}}'^{\mathrm{n}}$ (Remark 3.1.3). We conclude that the composed morphism $C \to \bar{C} \to \hat{\tilde{B}}'^{\mathrm{n}\natural} \to C'$ is compatible with f.

This proves Theorem 4.1.1 except for the factorization through f^+ (the kernel of $R^+ \to C'$ might not even be a prime above $\mathfrak{p}$) - which is already *enough to prove Theorem 1.1.1 in the mixed characteristic case*.

4.3.5. *Taking care of* f^+. We now fix a lifting $f^+ : R^+ \to R'^+$ of f (which exists by Proposition 2.1.1) and denote by $\mathfrak{p}^+$ its kernel (which is a prime of R^+ above $\mathfrak{p}$). In order to take f^+ into account, we will have to modify C' in the spirit of part (2) of Theorem 3.1.1. By the (usual) Abhyankar lemma, we may write R^+ as the directed colimit of normal finite sub-R-algebras R_β such that for some positive integer e_β, R_β contains $x_2^{\frac{1}{e_\beta}}$ and its ramification index is e_β at $\mathfrak{p}_\beta := \mathfrak{p}^+ \cap R_\beta$ (the latter condition amounts to: $R_\beta \subset K'[[x_2^{\frac{1}{e_\beta}}]]$). We note that R_β is also a complete local domain with residue field k, and that $\beta \leq \beta'$ implies $e_\beta \mid e_{\beta'}$.

We set $R'_\beta := R_\beta/\mathfrak{p}_\beta$. Then $R'^+ = R^+/\mathfrak{p}^+ = \cup R'_\beta$, and $R^+ \overset{f^+}{\twoheadrightarrow} R'^+$ is the colimit of the induced maps $R_\beta \overset{f_\beta}{\twoheadrightarrow} R'_\beta$. We can perform the previous steps with $(R_\beta, \mathfrak{p}_\beta)$ instead of $(R, \mathfrak{p})$. In step 4.3.1, we can replace $\underline{x}$ by the system of parameters $\underline{x}_\beta := (p, x_2^{\frac{1}{e_\beta}}, x_3, \ldots, x_d)$. In step 4.3.2, the same $\bar{C}$ works for every β. In step 4.3.3, we can choose an element $\pi_\beta \in R_\beta \setminus \mathfrak{p}_\beta$ divisible by p, such that $\pi_\beta \mathfrak{p}_\beta \subset x_2^{\frac{1}{e_\beta}} R_\beta$. In this way, the pairs $\underline{\beta} = (\beta, \pi_\beta)$ form a directed set (with the order $(\beta, \pi_\beta) \leq (\beta', \pi_{\beta'})$ if $\beta \leq \beta'$ and π_β divides $\pi_{\beta'}$).

We define the R'_β-algebra $B'_{\underline{\beta}} := {\pi_\beta}^{-\frac{1}{p^\infty}} \bar{C}$, and the $\tilde{R}'_\beta$-algebra $\tilde{B}'_{\underline{\beta}}$ as in part (1) of the proof of Theorem 3.1.1. The tilts $\widehat{\tilde{B}'_{\underline{\beta}}}^\flat$ form a directed system of $\widehat{\tilde{R}'_\beta}^\flat$-algebras, which are $(\pi_\beta^\flat)^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay $k[[p^\flat, x_3^\flat, \ldots, x_d^\flat]]$-algebras. Their ($p^\flat$-adically completed) colimit with respect to $\underline{\beta}$ maps to a $\hat{\mathcal{K}}^{\flat o}$-perfect(oid) Cohen-Macaulay $k[[p^\flat, x_3^\flat, \ldots, x_d^\flat]]$-algebra by Proposition 3.1.2 (2). Untilting and completing $\mathfrak{m}_{R'}$-adically, we get a perfectoid Cohen-Macaulay R'-algebra C' (Proposition 2.2.1), which is also an algebra over $\widehat{R'^+} = \widehat{\operatorname{colim}} \tilde{R}'_\beta$, and the composed morphism $C \to \bar{C} \to \operatorname{colim} \hat{\tilde{B}}^\natural_{\underline{\beta}} \to \operatorname{colim} \hat{\tilde{B}}_{\underline{\beta}} \to C'$ is compatible with $f^+ = \operatorname{colim} f_\beta$, by the same argument as in 4.3.4. $\square$

4.4. **Proof of Theorem 4.1.1: the case when R is of mixed characteristic and R' is of characteristic p.** The proof is similar (somewhat simpler), the roles of the first two parameters being switched. Now $R_\mathfrak{p}$ is a discrete valuation ring of mixed characteristic.

4.4.1. *Choosing an appropriate system of parameters of R.* By the Cohen-Gabber structure theorem (based on Epp's elimination of wild ramification) [16, IV 4.2], there exists a finite normal extension S of R in R^+ which becomes a finite extension of a ring $A := V[[x_2, \ldots, x_d]]$ (where V is a complete discrete valuation ring with residue field $k = k^+$), such that $A \to S$ becomes etale after inverting some element of A not contained in any minimal prime above p. Let x_1 be a uniformizer of V. One has $V[\frac{1}{p}] = V[\frac{1}{x_1}]$, and if e_0 denotes the absolute ramification index of V, $p^{-\frac{1}{e_0}} x_1$ is a unit in $W(k)^+$.

We may and shall replace R by S (and $\mathfrak{p}$ by a prime above it). Then $\underline{x} := (x_1, \ldots, x_d)$ is a system of parameters of R and x_1 is a uniformizer of the discrete valuation ring of mixed characteristic $R_\mathfrak{p}$. One has $R[\frac{1}{p}] = R[\frac{1}{x_1}]$, $A \cap \mathfrak{p} = x_1 A$, and the image of $(x_2, \ldots, x_d)$ in R' is a system of parameters of R'. We also fix a compatible sequence of roots $x_1^{\frac{1}{e}} \subset R^+$.

4.4.2. $\bar{C} := C/x_1^{\frac{1}{\infty}} C$ *is a perfect Cohen-Macaulay R/x_1-algebra (with respect to $\underline{x}'$).* Since C is actually perfectoid, Frobenius induces an isomorphism $C/p^{\frac{1}{pe_0 e}} \xrightarrow{\sim} C/p^{\frac{1}{e_0 e}}$ for any $e > 0$, hence an isomorphism $C/x_1^{\frac{1}{pe}} \xrightarrow{\sim} C/x_1^{\frac{1}{e}}$ since $p^{-\frac{1}{e_0}} x_1$ is a unit in C. Passing to the colimit on e, we conclude that $\bar{C}$ is perfect.

As in 4.3.2, C is faithfully flat over $V[x_1^{\frac{1}{e}}][[x_2, \ldots, x_d]]$, hence also over their colimit. Thus $\bar{C}$ is faithfully flat over $A' := k[[x_2, \ldots, x_d]]$.

4.4.3. *Constructing an almost Cohen-Macaulay R'-algebra B' with respect to $\underline{x}'$.* Since $R_\mathfrak{p}$ is a discrete valuation ring with uniformizer x_1, there exists $\pi \in R \setminus \mathfrak{p}$ such that

$$\pi\mathfrak{p} \subset x_1 R. \tag{4.6}$$

We fix a compatible system of p^{th}-power roots $\pi^{\frac{1}{p^i}}$ in R^+ and view C and $\bar{C}$ as $\mathbb{Z}[\pi^{\frac{1}{p^\infty}}]$-algebras. By (4.6), and since $\bar{C}$ is perfect, hence reduced, one has the inclusions

$$\pi^{\frac{1}{p^\infty}} \sqrt{\mathfrak{p}C} \subset \sqrt{x_1 C} \subset x_1^{\frac{1}{\infty}} C. \tag{4.7}$$

By the same argument as in 4.3.3 (involving Proposition 2.5.1 (2)), $\bar{C}$ is $\pi^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay with respect to $\underline{x}'$. It may have $\pi^{\frac{1}{p^\infty}}$-torsion, and as above, we introduce $B' := \pi^{-\frac{1}{p^\infty}} \bar{C}$, which is still $\pi^{\frac{1}{p^\infty}}$-almost Cohen-Macaulay with respect to $\underline{x}'$.

4.4.4. *Obtaining C' from Dietz's construction.* By Proposition 3.1.2 (1), the R'-algebra B' maps to a perfect, $\mathfrak{m}_{R'}$-adically separated, Cohen-Macaulay R'-algebra D' without zero-divisors. Denoting by C' its $\mathfrak{m}_{R'}$-adic completion, which is still a perfect Cohen-Macaulay R'-algebra, one gets a composed homomorphism $C \to \bar{C} \to B' \to D' \to C'$ compatible with f.

In contrast to the situation 4.3.4, the kernel of $R^+ \to C'$ is a prime $\mathfrak{p}^+$ lying above $\mathfrak{p}$ (since D' is a domain, and is $\mathfrak{m}_{R'}$-adically separated), so that $C \to C'$ factors through some lifting $f'^+ : R^+ \to R'^+$ of f. As R is normal (according to our preliminary reduction, *cf.* 4.1.1), one has $f'^+ := f \circ \sigma$ for some $\sigma \in G_K$ (Proposition 2.1.1 (2)), and one may precompose accordingly $R^+ \to C$ with σ. This proves Theorem 4.1.1 up to a Galois ambiguity on the factorization through f^+ - which is already *enough* (together with the previous steps) *to prove Theorem 1.1.1.*

4.4.5. *Taking care of f^+.* We proceed as in 4.3.5. We may write R^+ as the directed colimit of normal finite sub-R-algebras R_β, and write $R'_\beta := R_\beta/\mathfrak{p}_\beta$, its image in R'^+ via f^+. As in 4.4.1, we may assume that there is $x_{1\beta} \in W(k)^+ \cap R_\beta$ such that $p^{-\frac{1}{e_\beta}} x_{1\beta}$ is a unit for some positive integer e_β, and $x_{1\beta}$ is a uniformizer of $(R_\beta)_{\mathfrak{p}_\beta}$. We note that R_β is also a complete local domain with residue field k, and that $\beta \leq \beta'$ implies $x_{1\beta'} \mid x_{1\beta}$ in $W(k)^+$.

We can perform the previous steps with $(R_\beta, \mathfrak{p}_\beta)$ instead of $(R, \mathfrak{p})$, replacing $\underline{x}$ by the system of parameters $\underline{x}_\beta := (x_{1\beta}, x_2, \ldots, x_d)$. The same $\bar{C}$ works for every β. In step 4.4.3, we can choose an element $\pi_\beta \in R_\beta \setminus \mathfrak{p}_\beta$ such that $\pi_\beta \mathfrak{p}_\beta \subset x_{1\beta} R_\beta$. In this way, the pairs $\underline{\beta} = (\beta, \pi_\beta)$ form a directed set as in 4.3.5. The construction of B' in 4.4.3 is canonical and leads to a directed family $R'_\beta \to \pi_\beta^{-\frac{1}{p^\infty}}\bar{C}$. Taking the completed colimit leads to the desired morphism $C \to C'$, which is compatible with f^+. $\square$

This finishes the proof of Theorem 4.1.1 (and also the proof of Theorem 1.2.1). $\square$

4.5. By iterated application of Cohen factorizations, any finite sequence $R_0 \xrightarrow{f_1} R_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} R_n$ of local homomorphisms fits into a commutative diagram

$$
\begin{array}{ccccccc}
R_0 & \xrightarrow{f_1} & R_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & R_n \\
\downarrow & & \downarrow & & & & \downarrow \\
R_0^{\cdot} & \xrightarrow{f_1^{\cdot}} & R_1^{\cdot} & \xrightarrow{f_2^{\cdot}} & \cdots & \xrightarrow{f_n^{\cdot}} & R_n^{\cdot}
\end{array}
\tag{4.8}
$$

where the vertical maps are flat (the last one being $\mathrm{id}_{\mathrm{R_n}}$), and the $f_i^{\cdot}$ are surjective. Combining this with Theorem 4.1.1, one obtains the following generalization of 1.2.1:

4.5.1. **Theorem.** *Any finite sequence $R_0 \xrightarrow{f_1} R_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} R_n$ of local homomorphisms of complete Noetherian local domains, with R_0 of mixed characteristic $(0, p)$, fits into a commutative diagram*

$$
\begin{array}{ccccccc}
R_0 & \xrightarrow{f_1} & R_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & R_n \\
\downarrow & & \downarrow & & & & \downarrow \\
R_0^+ & \xrightarrow{f_1{}^+} & R_1^+ & \xrightarrow{f_2{}^+} & \cdots & \xrightarrow{f_n{}^+} & R_n^+ \\
\downarrow & & \downarrow & & & & \downarrow \\
C_0 & \longrightarrow & C_1 & \longrightarrow & \cdots & \longrightarrow & C_n
\end{array}
\tag{4.9}
$$

where each C_i is a $\hat{\mathcal{K}}^o$-perfectoid Cohen-Macaulay R_i-algebra if R_i is of mixed characteristic $(0, p)$, or a perfect Cohen-Macaulay R_i-algebra if R_i is of characteristic p.

Moreover, the f_i^+ can be given in advance. $\square$

Here $\hat{\mathcal{K}}^o$ is the perfectoid ring obtained from the Witt ring of the algebraic closure of the residue field of R_n by adjoining p^{th}-power roots of p and completing.

ERRATUM TO [1]

When almost algebra departs from its elementary categorical foundation (as a special case of Gabriel localization) to take its true nature, one may stumble on many subtle difficulties. We are grateful to O. Gabber for pointing out the following erratum to [1, §1][11].

Every almost projective $\mathfrak{A}$-module $\mathfrak{P}$ is flat [8, 2.4.12]. But it is not clear that it is almost *faithfully* flat if it is *faithful*; it is not even clear that every almost finite etale extension is almost *faithfully* flat (as written in [1, rem. 1.7.2 (1), 1.8.1. (1)])[12]. The results [8, 2.4.28 v, resp. iv] only show that the evaluation ideal $\mathcal{E}_{\mathfrak{P}/\mathfrak{A}} \subset \mathfrak{A}$ satisfies $\mathcal{E}_{\mathfrak{P}/\mathfrak{A}}\mathfrak{P} = \mathfrak{P}$ if $\mathfrak{P}$ is faithful, whereas *faithful* flatness amounts to $\mathcal{E}_{\mathfrak{P}/\mathfrak{A}} = \mathfrak{A}$; the difficulty is that it is not clear that the idempotent ideal $\mathcal{E}_{\mathfrak{P}/\mathfrak{A}}$ is generated by an idempotent almost-element.

However, the problem disappears if $\mathfrak{P}$ is of constant rank r because $\mathcal{E}_{\mathfrak{P}/\mathfrak{A}} \supset \mathcal{E}_{\wedge^r\mathfrak{P}/\mathfrak{A}} = \mathfrak{A}$ (*cf.* [8, proof of 4.3.8]), and more generally if $\mathfrak{P}$ is of finite rank.

---

[11]besides, as a referee pointed out, the expression "$\mathfrak{m}$ agit fidèlement sur $\mathcal{B}$" on the line following (2.10) in [1] should be replaced by "$\mathcal{B}$ est sans $\mathfrak{m}$-torsion".

[12]this is an important issue since the perfectoid Abhyankar lemma asserts that a certain extension is almost finite etale, whereas it is the almost faithful flatness of this extension which matters in the applications.

Fortunately, all almost finite etale extensions occurring in [1] are of finite rank as almost projective modules, because they occur, in the context of [1, Prop. 1.9.1 (1),(3)], in connection with Galois extensions and subextensions. However, in the given proof that Galois extensions are of constant rank $r = \mid G \mid$, the argument by faithful flatness descent should be replaced by the following observation (Gabber): for any extension $\mathfrak{A} \hookrightarrow \mathfrak{B}$ and any (almost) finite projective $\mathfrak{A}$-module $\mathfrak{P}$ such that $\mathfrak{P}_{\mathfrak{B}}$ is of rank r, $\mathfrak{P}$ is of rank r. Indeed, $\wedge^{r+1}\mathfrak{P}$ injects into $\wedge^{r+1}\mathfrak{P}_{\mathfrak{B}} = 0$, hence $\mathfrak{P}$ is of finite rank; furthermore, by [8, 4.3.27], $\mathfrak{A}$ decomposes into a finite product $\prod \mathfrak{A}_i$ where $\mathfrak{P}_{\mathfrak{A}_i}$ has rank i, and by tensoring with $\mathfrak{B}$, one sees that $\mathfrak{P}_{\mathfrak{A}_i} = 0$ for $i \neq r$.

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